Schrodinger's Wave Equation.

In 1926, Schrodinger using de-Broglie's idea of matter waves & developed a mathematical formula which is known as wave-mechanics derivation.

Let us consider the vibration of a stretched string.

If ω be the amplitude of any point whose co-ordinate is 'x' at time 't'

The appropriate form of the wave equation may be written as follows

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2} \frac{\partial^2 \omega}{\partial t^2} \quad \text{——— (i)}$$

where 'u' is the velocity of propagation of the wave on separating the variables, this differential equation may be written as

$$\omega = f(x) \cdot g(t) \quad \text{——— (ii)}$$

where $f(x)$ is a function of the co-ordinate x only and $g(t)$ is a function of time 't' only.

For the motion of standing waves such as occuring in a stretched string, it is possible to express $g(t)$ as

$$g(t) = A \sin \omega t = A \sin 2\pi \nu t \quad \text{— (iii)}$$

where ν is the vibrational frequency & A is constant and it stands for maximum amplitude.

from eq^n (ii) & (iii)

$$\omega = f(x) \cdot A \sin 2\pi\nu t$$

On differentiating the above equation with respect to 't' we have,

$$\frac{\partial \omega}{\partial t} = f(x) \cdot A \cos 2\pi\nu t \cdot 2\pi\nu$$

$$\frac{\partial^2 \omega}{\partial t^2} = f(x) (-) A \sin 2\pi\nu t \cdot 2\pi\nu \cdot 2\pi\nu$$

or $$\frac{\partial^2 \omega}{\partial t^2} = -4\pi^2\nu^2 f(x) \cdot A \sin 2\pi\nu t$$

$[\because A \sin 2\pi\nu t = g(t)]$

$$= -4\pi^2\nu^2 f(x)\, g(t) \quad \text{——— (iv)}$$

Now differentiating the eq^n (ii) with respect to x

$$\omega = f(x) \cdot g(t)$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial f(x)}{\partial x} \cdot g(t)$$

or $$\frac{\partial^2 \omega}{\partial x^2} = \frac{\partial^2 f(x)}{\partial x^2} \cdot g(t) \quad \text{——— (v)}$$

From equation (iv) and (v) i.e substituting the value of $\frac{\partial^2 \omega}{\partial t^2}$ and $\frac{\partial^2 \omega}{\partial x^2}$, in equation (i) i.e $\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2} \cdot \frac{\partial^2 \omega}{\partial t^2}$

$$\frac{\partial^2 f(x)}{\partial x^2} \cdot g(t) = \frac{1}{u^2} \left(-4\pi^2 \nu^2\right) f(x) \cdot g(t)$$

$$\text{or} \quad \frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2 \nu^2}{u^2} f(x) \text{ --- (vi)}$$

But ν and u are related by the equation $u = \nu\lambda$ or $u^2 = \nu^2\lambda^2$, on Putting this value in the above eq^n we have.

$$\therefore \quad \frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2 \cancel{\nu^2}}{\cancel{\nu^2}\lambda^2} f(x) \text{ ---}$$

$$\frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} f(x) \text{ --- (vii)}$$

It is the expression for the wave eq^n in one direction and it can be extended in three directions, expressed by the co-ordinate x, y and z. If $f(x)$ for one co-ordinate is replaced by three co-ordinate x, y & z i.e $\psi(x, y z)$, which is amplitude function for three co-ordinates. then eq^n (vii) takes the form as follows.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2} \psi \text{ --- (viii)}$$

Using the symbol ∇^2 for differential Operator

$$\text{i.e} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Here ∇^2 (Del squared) is known as Laplacian operator.

then eq^n (viii) may be replaced by

$$\nabla^2 \psi = -\frac{4\pi^2}{\lambda^2} \psi \text{ --- (ix)}$$

The above treatment is applicable to all particles including electrons, atoms and Photons.

By using, de-Broglie's relation $\lambda = \frac{h}{mu}$ in the eq^n (ix)

we have

$$\nabla^2 \psi = -\frac{4\pi^2}{h^2} m^2 u^2 \psi \text{ --- (x)}$$

where m is mass of Particle, u is velocity and h is Plank Constant.

be Continued - - - -

But we know that the total Energy of a Particle is the sum of its Potential energy and Kinetic energy

If, E = Total energy, U = Potential energy and Kinetic energy = $\frac{1}{2}mu^2$

$\therefore$ $E = K.E + P.E$

$E = \frac{1}{2}mu^2 + U$

or $2(E-U) = mu^2$ ———— (xi)

On Substituting the value of mu^2 in eqn (x)

we have $\nabla^2\psi = -\frac{4\pi^2}{h^2} m \cdot 2(E-U) \cdot \psi$

or $\nabla^2\psi = -\frac{8\pi^2}{h^2} m \; (E-U)\psi$

or $\nabla^2\psi + \frac{8\pi^2 m}{h^2}(E-U)\psi = 0$ —(xii)

It is the required form of Schroedinger's Wave Equation.

Schrodinger's Wave Equation for Hydrogen atom.

We know that the Potential Energy of H atom $= -\frac{e^2}{r}$

and General form of Schrodinger's wave equation is given as

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}(E-U)\psi = 0$$

where E is total energy and U is Potential energy

$$\text{or,}\quad \nabla^2\psi + \frac{8\pi^2 m}{h^2}\left\{[E] - \left(-\frac{e^2}{r}\right)\right\}\psi = 0$$

$$\text{or,}\quad -\frac{h^2}{8\pi^2 m}\nabla^2\psi = E\psi + \frac{e^2}{r}\psi$$

$$\text{or}\quad E\psi = \left[-\frac{h^2}{8\pi^2 m}\left(\nabla^2\psi + \frac{e^2}{r}\right)\psi\right]$$

$$E\psi = \left[-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{e^2}{r}\right]\psi$$

The above eqn is the Schrodinger's wave equation for Hydrogen atom.

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